

CONSTRUCTIVE PROOF OF BROUWER'S FIXED POINT THEOREM FOR SEQUENTIALLY LOCALLY NON-CONSTANT FUNCTIONS

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ABSTRACT. We present a constructive proof of Brouwer's fixed point theorem for uniformly continuous and sequentially locally non-constant functions based on the existence of approximate fixed points. And we will show that Brouwer's fixed point theorem for uniformly continuous and sequentially locally non-constant functions implies Sperner's lemma for a simplex. Since the existence of approximate fixed points is derived from Sperner's lemma, our Brouwer's fixed point theorem is equivalent to Sperner's lemma.

1. INTRODUCTION

It is well known that Brouwer's fixed point theorem can not be constructively proved¹. Sperner's lemma which is used to prove Brouwer's theorem, however, can be constructively proved. Some authors have presented an approximate version of Brouwer's theorem using Sperner's lemma. See [8] and [9]. Thus, Brouwer's fixed point theorem is constructively, in the sense of constructive mathematics à la Bishop, proved in its approximate version.

Also [8] states a conjecture that a uniformly continuous function f from a simplex into itself, with property that each open set contains a point x such that $x \neq f(x)$, which means $|x - f(x)| > 0$, and also at every point x on the boundaries of the simplex $x \neq f(x)$, has an exact fixed point. We call such a property of functions *local non-constancy*. In this note we present a partial answer to Dalen's conjecture.

Recently [2] showed that the following theorem is equivalent to Brouwer's fan theorem.

Each uniformly continuous function f from a compact metric space X into itself with at most one fixed point and approximate fixed points has a fixed point.

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¹[6] provided a *constructive* proof of Brouwer's fixed point theorem. But it is not constructive from the view point of constructive mathematics à la Bishop. It is sufficient to say that one dimensional case of Brouwer's fixed point theorem, that is, the intermediate value theorem is non-constructive. See [3] or [8].

By reference to the notion of *sequentially at most one maximum* in [1] we require a condition that a function f is *sequentially locally non-constant*, and will show the following result.

Each uniformly continuous function f from an n -dimensional simplex into itself, which is sequentially locally non-constant and has approximate fixed points, has a fixed point,

without the fan theorem. Sequential local non-constancy is stronger than the condition in [8] (local non-constancy), and is different from the condition that a function has *at most one fixed point* in [1].

[7] constructed a computably coded continuous function f from the unit square into itself, which is defined at each computable point of the square, such that f has no computable fixed point. His map consists of a retract of the computable elements of the square to its boundary followed by a rotation of the boundary of the square. As pointed out by [5], since there is no retract of the square to its boundary, his map does not have a total extension.

In the next section we present our theorem and its proof. In Section 3 we will derive Sperner's lemma from Brouwer's fixed point theorem for uniformly continuous and sequentially locally non-constant functions.

2. BROUWER'S FIXED POINT THEOREM FOR SEQUENTIALLY LOCALLY NON-CONSTANT FUNCTIONS

We consider an n -dimensional simplex Δ as a compact metric space. Let x be a point in Δ , and consider a uniformly continuous function f from Δ into itself.

According to [8] and [9] f has an approximate fixed point. It means

$$\text{For each } \varepsilon > 0 \text{ there exists } x \in \Delta \text{ such that } |x - f(x)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$\inf_{x \in \Delta} |x - f(x)| = 0.$$

The definition of local non-constancy of functions in a simplex Δ is as follows;

Definition 1. (*Local non-constancy of function*)

- (1) *At a point x on the faces (boundaries) of Δ $f(x) \neq x$. This means $f_i(x) > x_i$ or $f_i(x) < x_i$ for at least one i , where x_i and $f_i(x)$ denote the i -th components of x and $f(x)$.*
- (2) *And in any open set in Δ there exists a point x such that $f(x) \neq x$.*

On the other hand the notion that f has at most one fixed point by [2] is defined as follows;

Definition 2 (At most one fixed point). *For all $x, y \in \Delta$, if $x \neq y$, then $f(x) \neq x$ or $f(y) \neq y$.*

By reference to the notion of *sequentially at most one maximum* in [1], we define the property of *sequential local non-constancy*.

First we recapitulate the compactness (total boundedness with completeness) of a set in constructive mathematics. Δ is totally bounded in the sense that for each $\varepsilon > 0$ there exists a finitely enumerable ε -approximation to Δ^2 . An ε -approximation

²A set S is finitely enumerable if there exist a natural number N and a mapping of the set $\{1, 2, \dots, N\}$ onto S .

to Δ is a subset of Δ such that for each $x \in \Delta$ there exists y in that ε -approximation with $|x - y| < \varepsilon$. According to Corollary 2.2.12 of [4] we have the following result.

Lemma 1. *For each $\varepsilon > 0$ there exist totally bounded sets H_1, H_2, \dots, H_n , each of diameter less than or equal to ε , such that $\Delta = \bigcup_{i=1}^n H_i$.*

Since $\inf_{x \in \Delta} |x - f(x)| = 0$, we have $\inf_{x \in H_i} |x - f(x)| = 0$ for some $H_i \subset \Delta$.

The definition of sequential local non-constancy is as follows;

Definition 3. *(Sequential local non-constancy of functions) There exists $\bar{\varepsilon} > 0$ with the following property. For each $\varepsilon > 0$ less than or equal to $\bar{\varepsilon}$ there exist totally bounded sets H_1, H_2, \dots, H_m , each of diameter less than or equal to ε , such that $\Delta = \bigcup_{i=1}^m H_i$, and if for all sequences $(x_n)_{n \geq 1}, (y_n)_{n \geq 1}$ in each H_i , $|f(x_n) - x_n| \rightarrow 0$ and $|f(y_n) - y_n| \rightarrow 0$, then $|x_n - y_n| \rightarrow 0$.*

Now we show the following lemma, which is based on Lemma 2 of [1].

Lemma 2. *Let f be a uniformly continuous function from Δ into itself. Assume $\inf_{x \in H_i} |f(x) - x| = 0$ for some $H_i \subset \Delta$ defined above. If the following property holds:*

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $x, y \in H_i$, $|f(x) - x| < \varepsilon$ and $|f(y) - y| < \varepsilon$, then $|x - y| \leq \delta$.

Then, there exists a point $z \in H_i$ such that $f(z) = z$, that is, f has a fixed point.

Proof. Choose a sequence $(x_n)_{n \geq 1}$ in H_i such that $|f(x_n) - x_n| \rightarrow 0$. Compute N such that $|f(x_n) - x_n| < \varepsilon$ for all $n \geq N$. Then, for $m, n \geq N$ we have $|x_m - x_n| \leq \delta$. Since $\delta > 0$ is arbitrary, $(x_n)_{n \geq 1}$ is a Cauchy sequence in H_i , and converges to a limit $z \in H_i$. The continuity of f yields $|f(z) - z| = 0$, that is, $f(z) = z$. \square

Next we show the following theorem, which is based on Proposition 3 of [1].

Theorem 1. *Each uniformly continuous function f from an n -dimensional simplex into itself, which is sequentially locally non-constant and has approximate fixed points, has a fixed point.*

Proof. Choose a sequence $(z_n)_{n \geq 1}$ in H_i defined above such that $|f(z_n) - z_n| \rightarrow 0$. In view of Lemma 2 it is enough to prove that the following condition holds.

For each $\delta > 0$ there exists $\varepsilon > 0$ such that if $x, y \in H_i$, $|f(x) - x| < \varepsilon$ and $|f(y) - y| < \varepsilon$, then $|x - y| \leq \delta$.

Assume that the set

$$K = \{(x, y) \in H_i \times H_i : |x - y| \geq \delta\}$$

is nonempty and compact³. Since the mapping $(x, y) \rightarrow \max(|f(x) - x|, |f(y) - y|)$ is uniformly continuous, we can construct an increasing binary sequence $(\lambda_n)_{n \geq 1}$ such that

$$\lambda_n = 0 \Rightarrow \inf_{(x, y) \in K} \max(|f(x) - x|, |f(y) - y|) < 2^{-n},$$

$$\lambda_n = 1 \Rightarrow \inf_{(x, y) \in K} \max(|f(x) - x|, |f(y) - y|) > 2^{-n-1}.$$

It suffices to find n such that $\lambda_n = 1$. In that case, if $|f(x) - x| < 2^{-n-1}$, $|f(y) - y| < 2^{-n-1}$, we have $(x, y) \notin K$ and $|x - y| \leq \delta$. Assume $\lambda_1 = 0$. If $\lambda_n = 0$, choose $(x_n, y_n) \in K$ such that $\max(|f(x_n) - x_n|, |f(y_n) - y_n|) < 2^{-n}$, and if $\lambda_n = 1$, set $x_n = y_n = z_n$. Then, $|f(x_n) - x_n| \rightarrow 0$ and $|f(y_n) - y_n| \rightarrow 0$, so $|x_n - y_n| \rightarrow 0$.

³See Theorem 2.2.13 of [4].

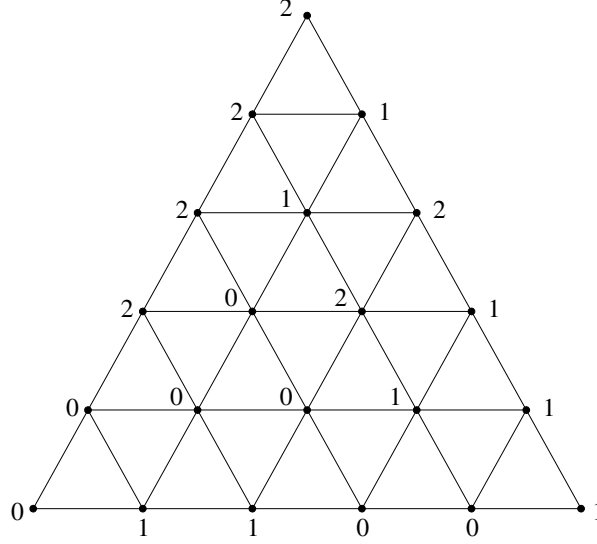


FIGURE 1. Partition and labeling of 2-dimensional simplex

Computing N such that $|x_N - y_N| < \delta$, we must have $\lambda_N = 1$. We have completed the proof. \square

3. FROM BROUWER'S FIXED POINT THEOREM FOR SEQUENTIALLY LOCALLY NON-CONSTANT FUNCTIONS TO SPERNER'S LEMMA

In this section we will derive Sperner's lemma from Brouwer's fixed point theorem for uniformly continuous and sequentially locally non-constant functions. Let Δ be an n -dimensional simplex. Denote a point on Δ by x . Consider a function f from Δ into itself. Partition Δ in the way depicted in Figure 1 for a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ . Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the above mentioned way, and draw the planes parallel to the faces of Δ . Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension. Let K denote the set of small n -dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

- (1) The vertices of Δ are respectively labeled with 0 to n . We label a point $(1, 0, \dots, 0)$ with 0 , a point $(0, 1, 0, \dots, 0)$ with 1 , a point $(0, 0, 1, \dots, 0)$ with $2, \dots$, a point $(0, \dots, 0, 1)$ with n . That is, a vertex whose k -th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k for all $k \in \{0, 1, \dots, n\}$.
- (2) If a vertex of a simplex of K is contained in an $n - 1$ -dimensional face of Δ , then this vertex is labeled with some number which is the same as the number of a vertex of that face.

- (3) If a vertex of a simplex of K is contained in an $n - 2$ -dimensional face of Δ , then this vertex is labeled with some number which is the same as the number of a vertex of that face. And similarly for cases of lower dimension.
- (4) A vertex contained inside of Δ is labeled with an arbitrary number among $0, 1, \dots, n$.

Denote the vertices of an n -dimensional simplex of K by x^0, x^1, \dots, x^n , the j -th coordinate of x^i by x_j^i , and denote the label of x^i by $l(x^i)$. Let τ be a positive number which is smaller than $x_{l(x^i)}^i$ for all x^i , and define a function $f(x^i)$ as follows⁴:

$$f(x^i) = (f_0(x^i), f_1(x^i), \dots, f_n(x^i)),$$

and

$$f_j(x^i) = \begin{cases} x_j^i - \tau & \text{for } j = l(x^i), \\ x_j^i + \frac{\tau}{n} & \text{for } j \neq l(x^i). \end{cases} \quad (1)$$

f_j denotes the j -th component of f . From the labeling rules we have $x_{l(x^i)}^i > 0$ for all x^i , and so $\tau > 0$ is well defined. Since $\sum_{j=0}^n f_j(x^i) = \sum_{j=0}^n x_j^i = 1$, we have

$$f(x^i) \in \Delta.$$

We extend f to all points in the simplex by convex combinations on the vertices of the simplex. Let z be a point in the n -dimensional simplex of K whose vertices are x^0, x^1, \dots, x^n . Then, z and $f(z)$ are expressed as follows;

$$z = \sum_{i=0}^n \lambda_i x^i, \text{ and } f(z) = \sum_{i=0}^n \lambda_i f(x^i), \lambda_i \geq 0, \sum_{i=0}^n \lambda_i = 1.$$

It is clear that f is uniformly continuous. We verify that f is sequentially locally non-constant.

- (1) Let z be a point in an n -dimensional simplex δ^n . Assume that no vertex of δ^n is labeled with i . Then

$$f_i(z) = \sum_{j=0}^n \lambda_j f_i(x^j) = z_i + \left(1 + \frac{1}{n}\right) \tau. \quad (2)$$

Then, there exists no sequence $(z_m)_{m \geq 1}$ such that $|f(z_m) - z_m| \rightarrow 0$ in δ^n .

- (2) Assume that z is contained in a fully labeled n -dimensional simplex δ^n , and rename vertices of δ^n so that a vertex x^i is labeled with i for each i . Then,

$$f_i(z) = \sum_{j=0}^n \lambda_j f_i(x^j) = \sum_{j=0}^n \lambda_j x_j^j + \sum_{j \neq i} \lambda_j \frac{\tau}{n} - \lambda_i \tau = z_i + \left(\frac{1}{n} \sum_{j \neq i} \lambda_j - \lambda_i \right) \tau \text{ for each } i.$$

Consider sequences $(z_m)_{m \geq 1}, (z'_m)_{m \geq 1}$ such that $|f(z_m) - z_m| \rightarrow 0$ and $|f(z'_m) - z'_m| \rightarrow 0$.

Let $z_m = \sum_{i=0}^n \lambda(m)_i x^i$ with $\lambda(m)_i \geq 0$, $\sum_{i=0}^n \lambda(m)_i = 1$ and $z'_m = \sum_{i=0}^n \lambda'(m)_i x^i$ with $\lambda'(m)_i \geq 0$, $\sum_{i=0}^n \lambda'(m)_i = 1$. Then, we have

$$\frac{1}{n} \sum_{j \neq i} \lambda(m)_j - \lambda(m)_i \rightarrow 0, \text{ and } \frac{1}{n} \sum_{j \neq i} \lambda'(m)_j - \lambda'(m)_i \rightarrow 0 \text{ for all } i.$$

⁴We refer to [10] about the definition of this function.

Therefore, we obtain

$$\lambda(m)_i \longrightarrow \frac{1}{n+1}, \text{ and } \lambda'(m)_i \longrightarrow \frac{1}{n+1}.$$

These mean

$$|z_m - z'_m| \longrightarrow 0.$$

Thus, f is sequentially locally non-constant, and it has a fixed point. Let z^* be a fixed point of f . We have

$$z_i^* = f_i(z^*) \text{ for all } i. \quad (3)$$

Suppose that z^* is contained in a small n -dimensional simplex δ^* . Let x^0, x^1, \dots, x^n be the vertices of δ^* . Then, z^* and $f(z^*)$ are expressed as

$$z^* = \sum_{i=0}^n \lambda_i x^i \text{ and } f(z^*) = \sum_{i=0}^n \lambda_i f(x^i), \quad \lambda_i \geq 0, \quad \sum_{i=0}^n \lambda_i = 1.$$

(1) implies that if only one x^k among x^0, x^1, \dots, x^n is labeled with i , we have

$$f_i(z^*) = \sum_{j=0}^n \lambda_j f_i(x^j) = \sum_{j=0}^n \lambda_j x_i^j + \sum_{j \neq k} \lambda_j \frac{\tau}{n} - \lambda_k \tau = z_i^* \quad (z_i^* \text{ is the } i\text{-th coordinate of } z^*).$$

This means

$$\frac{1}{n} \sum_{j \neq k} \lambda_j - \lambda_k = 0.$$

Then, (3) is satisfied with $\lambda_k = \frac{1}{n+1}$ for all k . If no x^j is labeled with i , we have (2) with $z = z^*$ and then (3) can not be satisfied. Thus, one and only one x^j must be labeled with i for each i . Therefore, δ^* must be a fully labeled simplex, and so the existence of a fixed point of f implies the existence of a fully labeled simplex.

We have completely proved Sperner's lemma.

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